STABILITY IN THE WEAK VARIATIONAL PRINCIPLE OF BAROTROPIC FLOWS

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Abstract

I find conditions under which the "Weak Energy Principle" of Katz, Inagaki and Yahalom (1993) gives necessary and sufficient conditions. My conclusion is that, necessary and sufficient conditions of stability are obtained when we have only two mode coupling in the gyroscopic terms of the perturbed Lagrangian. To illustrate the power of this new energy principle, I have calculated the stability limits of two dimensional configurations such as ordinary Maclaurin disk, an infinite self gravitating rotating sheet, and a two dimensional Rayleigh flow which has well known sufficient conditions of stability. All perturbations considered are in the same plane as the configurations. The limits of stability are identical with those given by a dynamical analysis when available, and with the results of the strong energy principle analysis when given. Thus although the "Weak Energy" method is mathematically more simple than the "Strong Energy" method of Katz, Inagaki and Yahalom)1993) since it does not involve solving second order partial differential equations, it is by no means less effective.

Key words: Energy variational principle; Self-gravitating systems; Stability of fluids.

I. Introduction

The main purpose of this work is to study the stability features of weak variational principle of barotropic fluid dynamics (Katz, Inagaki and Yahalom (1993) from now on paper I). This principle has the advantage that all equations of barotropic fluid dynamics are derived from one Lagrangian, i.e. Euler equations and continuity equation. And therefore stability analysis in this formalism does not involve solving partial differential equations like in the dynamical perturbation method or the strong variational principle (see paper I). However, it has the disadvantage of lacking standard form. This Lagrangian contain only a term depending on the degrees of freedom (a potential) and a term linear in time derivatives (gyroscopic term), but lacks a term quadratic in time derivatives (kinetic term). The above fact is the cause of some novel stability features I discuss below.

The plan of this paper is as follows, section II contains a quick introduction to the weak variational principle of barotropic flows given in (paper I), some modifications are introduced for both 3-D and 2-D cases . In section III I overview the stability theory of a system which does not contain terms quadratic in time derivatives (gyroscopic system), a comparison between stability predictions using the strong variational principle and weak variational principle is also given.

In section IV I introduce the general form of the second variation of a barotropic flow potentail. Later a few illustrations of stability analysis using the weak energy principle in two dimensional flows is given. In section V I give a general formalism of uniformly rotating galactic flows. In section VI I study a specific case of the above flows, the uniform rotating sheet which was first analyzed by Binney J. & Tremaine S. (1987). In section VII I

analyse the stability of another galactic model known as the "Maclaurin disks" which also appears at Binney J. & Tremaine S. (1987). This model is analysed dynamicaly in (Binney J. & Tremaine S. (1987)) by solving three equations (Euler equations in two dimensions and the continuity equation). It is also analysed in (Yahalom, Katz & Inagaki 1994 from now on paper II) using the strong energy method and solving one equation (continuity equation). Here it is analysed without solving any equation, but merely diaganolizing the appropriate potential. In section VIII I analyse the two dimensional Rayleigh flows , where sufficient conditions of stabilty were found by Lord Rayleigh (1880).

II. The Weak Variational Principle

The weak variational principle is derived in paper I from the Lagrangian (5.16) combined with (6.12) :

$$L = \int [\vec{w} \cdot \vec{v} - \left(\frac{1}{2}\vec{v}^2 + \varepsilon + \frac{1}{2}\Phi\right)]\rho d^3x + \vec{b} \cdot (\vec{P} - \vec{P}_0) + \vec{\Omega}_c \cdot (\vec{J} - \vec{J}_0) \qquad (II.1)$$

We use the following notations: for the positions of fluid elements $\vec{r}$ or $(x^K) = (x, y, z)$; $K, L, ... = 1, 2, 3$ the density of matter is ρ; $\vec{v}$ is the velocity field in inertial coordinates. $\vec{b}$ and $\vec{\Omega}_c$ serve as Lagrange multipliers with respect to linear and angular momenta respectively given by:

$$\vec{P} = \int \vec{v}\rho d^3x \qquad (II.2.a)$$

and

$$\vec{J} = \int \vec{r} \times \vec{v}\rho d^3x \qquad (II.2.b)$$

$\varepsilon(\rho)$ is the specific internal energy of the barotropic fluid, related to the pressure and the

specific enthalpy:

$$\varepsilon(\rho) = h - \frac{P}{\rho} \qquad P = \rho^2 \frac{\partial \varepsilon}{\partial \rho} \tag{$II.3$}$$

Φ is the internal gravitational potential given by:

$$\Phi = -G \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3x'. \tag{$II.4$}$$

In our formalism $\vec{v}$ has a Clebsch form:

$$\vec{v} = \alpha \vec{\nabla} \beta + \vec{\nabla} \nu. \tag{$II.5$}$$

And ρ is given by:

$$\rho = \frac{\partial(\alpha, \beta, \mu)}{\partial(x, y, z)}. \tag{$II.6$}$$

$\vec{w}$ is defined by:

$$\dot{\alpha} + \vec{w} \cdot \vec{\nabla} \alpha = 0 \qquad \dot{\beta} + \vec{w} \cdot \vec{\nabla} \beta = 0 \qquad \dot{\mu} + \vec{w} \cdot \vec{\nabla} \mu = 0. \tag{$II.7$}$$

Hence our system is described by four trial functions α, β, μ, ν and two Lagrange multipliers $\vec{b}$ and $\vec{\Omega}_c$. Taking the variation of Lagrangian (II.1) with respect to the four trial functions gives the Euler and mass conservation equations in moving coordinates. Taking the variation of Lagrangian (II.1) with respect to the two Lagrange multipliers gives the fixation of linear and angular momentum. Taking the variation of the potential part of (II.1) gives the equation of a stationary barotropic fluid. Further details can be found in paper I. I will know modify the Lagrangian slightly. Rewriting the first part of equation (II.1) using equation (II.5) :

$$\int \vec{w} \cdot \vec{v} \rho d^3x = \int \vec{w} \cdot (\alpha \vec{\nabla} \beta + \vec{\nabla} \nu) \rho d^3x = \int (-\alpha \dot{\beta} \rho - \nu \vec{\nabla} \cdot (\rho \vec{w})) d^3x \tag{$II.8$}$$

where in the second equality we integrated by parts and neglected boundary terms (we also assumed that ν is single valued) and used equation (II.7). Thus we obtain:

$$\int \vec{w} \cdot \vec{v} \rho d^3x = \int (-\alpha \dot{\beta} \rho + \nu \dot{\rho}) d^3x = \int (-\alpha \dot{\beta} - \dot{\nu}) \rho d^3x \qquad (II.9)$$

where in the second equality we added a full time derivative. Inserting equation (II.9) into equation (II.1) and using equation (II.5) we obtain:

$$L = \int \left[-\alpha \dot{\beta} - \dot{\nu} - \frac{1}{2} (\alpha \vec{\nabla} \beta + \vec{\nabla} \nu)^2 + \varepsilon + \frac{1}{2} \Phi \right] \rho d^3x + \vec{b} \cdot (\vec{P} - \vec{P}_0) + \vec{\Omega}_c \cdot (\vec{J} - \vec{J}_0). \qquad (II.10)$$

In 2-D flows the formalism is modified slightly. Euler equations have only two components, linear momentum has two components $\vec{P}$. Angular momenntum J and vorticity $\omega = \vec{\nabla} \times \vec{v} \cdot \vec{1}_z$ have one component. The density ρ is given by:

$$\rho = \Sigma \delta_D(z) = \lambda(\alpha) \frac{\partial(\alpha, \beta)}{\partial(x, y)} \delta_D(z). \qquad (II.11)$$

δ_D is diracs delta, Σ is the surface density, and λ is a function of α depending on the flow under consideration. The internal energy ε and pressure P are considred to be functions of Σ. The Lagrangian of a 2-D flow does not depend on μ and is given by:

$$L = \int \left[-\alpha \dot{\beta} - \dot{\nu} - \frac{1}{2} (\alpha \vec{\nabla} \beta + \vec{\nabla} \nu)^2 + \varepsilon + \frac{1}{2} \Phi \right] \Sigma d^2x + \vec{b} \cdot (\vec{P} - \vec{P}_0) + \Omega_c (J - J_0). \qquad (II.12)$$

For Further details see paper II.

III. Stability of Gyroscopic Systems

It is our purpose now to derive from the functional form of the potential term in the Lagrangian (II.10) the conditions under which a given stationary configuration becomes stable or unstable. To do this we study a system with N degrees of freedom that is described by the Lagrangian:

$$L = G - V = b_i(q) \dot{q}^i - V(q) \qquad (III.1)$$

(a summation agreement is assumed) the energy will become

$$E = V(q). \tag{III.2}$$

This means that the system can propogate only on a manifold of equipotential, if the equipotential manifold is zero dimensional as in the case of a maximum or a minimum of the potential than the system is bound to remain on this point i.e. we have a sufficient condition of stability. This can also be deduced from the following argument, suppose we distort our equilibrium configuration given by $q^i = q^i_0, \dot{q}^i_0 = 0$ and $\frac{\partial V}{\partial q_i}|_{q0} = 0$ slightly such that initial conditions and energy are slightly different. Since this is done around equilibrium there is no first order contribution to the variation of energy and we obtain:

$$\delta^2 E = \frac{\partial^2 V}{\partial q_i \partial q_j}|_{q0} \delta q_i \delta q_j \equiv V_{ij} \delta q_i \delta q_j. \tag{III.3}$$

From equation (III.3) it is easy to see that if the matrix V_{ij} has a definite sign (i.e. we have a minimum or a maximum) , the system is bounded (i.e. stable). Let us now take the dynamical point of view, writing down the linearized Euler-Lagrange equations of this system we obtain:

$$V_{ij} \delta q_i = (\frac{\partial b_i}{\partial q_j} - \frac{\partial b_j}{\partial q_i})|_{q0} \delta \dot{q}_i \equiv b_{ij} \delta \dot{q}_i. \tag{III.4.a}$$

The above expression can be derived from the perturbed Lagrangian:

$$\delta^2 L = \delta^2 G - \delta^2 V = b_{ij} \delta q_i \delta \dot{q}_j - V_{ij} \delta q_i \delta q_j \tag{III.4.b}$$

Now suppose that:

$$\delta q_i \propto e^{\omega t} \tag{III.5}$$

this means that:

$$V_{ij}\delta q_i = \omega b_{ij}\delta q_i. \qquad (III.6)$$

in order to obtain the eigen frequencies we must solve the equation.

$$det|V_{ij} - \omega b_{ij}| = 0. \qquad (III.7)$$

Assume without the loss of generality that V_{ij} is diaganolized. Taking a two dimensional perturbation (a two mode coupling in $\delta^2 G$) we have:

$$\omega^2 = -\frac{V_{11}V_{22}}{b_{12}^2} \qquad (III.8)$$

in this case a definite sign is a necessary and sufficient condition of stability as can be clearly seen from equation (III.8). Moving on to the third dimensional case:

$$\omega^2 = -\frac{V_{11}V_{22}V_{33}}{b_{12}^2 V_{33} + b_{13}^2 V_{22} + b_{23}^2 V_{11}} \qquad (III.9)$$

here a definite sign is not necessary, take for example:

$$V_{11} < 0 \ V_{22} < 0 \ V_{33} > 0 \qquad \text{with} \qquad b_{12}^2 V_{33} + b_{13}^2 V_{22} + b_{23}^2 V_{11} > 0. \qquad (III.10)$$

Thus we conclude that a neseccary and sufficient condition of stability appear only in the case where the gyroscopic term couple only two modes, for more complex coupling the energy criterion is only sufficient.

III.1 Strong systems derived from Gyroscopic systems

In this section we diverge from the main topic of this paper in order to study the relations between weak and strong variational principles (paper I). Observing closely the

Lagrangian (II.10) we see that it contains two kinds of variables, i.e. we have $N+M$ variables q_i, ν_a (we take $i,j,k \in [1..N]$ and $a,b,c \in [1..M]$) , the Lagrangian has the following structure:

$$L_w = b_i^1(q)\dot{q}_i + b_a^2(q)\dot{\nu}_a - V^w(q,\nu), \qquad V^w = V^0(q) + V_a^1(q)\nu_a + \frac{1}{2}\nu_a\nu_a \qquad (III.11)$$

The equations of motion of this system are:

$$b^1_{[i,k]}\dot{q}_k - b^2_{a,i}\dot{\nu}_a = -V^w_{,i} \qquad (III.12.a)$$

$$\nu_a = -b^2_{a,i}\dot{q}_i - V^1_a. \qquad (III.12.b)$$

Using equation (III.12.b) we eliminate ν_a in (III.11) and obtain:

$$L_s = \frac{1}{2}g_{ij}\dot{q}_i\dot{q}_j + b^s_i\dot{q}_i - V^s, \qquad g_{ij} = b^2_{a,i}b^2_{a,j}, \qquad b^s_i = b^1_i + V^1_a b^2_{a,i}, \qquad V^s = V^0 - \frac{1}{2}V^1_a V^1_a. \qquad (III.13)$$

The euilibriums of the two systems are the same. In equilibrium $\nu_{a0} = -V^1_a$ (see (III.12.b)), inserting this into equation (III.12.a) we obtain:

$$V^w_{,i} = (V^0 - \frac{1}{2}V^1_a V^1_a)_{,i} = V^s_{,i} = 0. \qquad (III.14)$$

Without loss of generality we consider the stability of an equilibrium given by $\nu_{a0} = 0, q_{i0}$, where q_{i0} are defined by equation (III.14).

The energy criterion of the two systems is also the same: For a quadratic system such as L_s the energy criterion takes the form $V^s_{,ij}|_0\delta q_i\delta q_j > 0$, while for a gyroscopic system such as L_w we demand that $\Delta^2 V^w = (\delta\nu_a + V^1_{a,i}\delta q_i)^2 + (V^0_{,ij} - V^1_{a,i}V^1_{a,j})\delta q_i\delta q_j = (\delta\nu_a + V^1_{a,i}\delta q_i)^2 + V^s_{,ij}|_0\delta q_i\delta q_j$ has a definite sign, i.e. $V^s_{,ij}|_0\delta q_i\delta q_j > 0$.

It is not to be understood that transforming the weak Lagrangian into L_s has no benefit from the point of view of stability analysis. In fact the strong principle provides us with necessary and sufficient conditions beyond what we obtain from (III.8). Every perturbation of L_s implicitly includes all $\delta\nu$ perturbations, that is we obtain necessary and sufficient conditions for more general perturbations than are obtained in the two mode analysis of equation (III.8). For example for a single mode, say δq_1 of L_s the gyroscopic term vanishes and we have a necessary and sufficient condition of stability:

$$V^s_{,11} = V^0_{,11} - V^1_{a,1} V^1_{a,1} > 0 \qquad (III.15.a)$$

this can be compared to a two mode analysis of L_w, with δq_1 and $\delta\nu_2$ for which we obtain:

$$V^0_{,11} - V^1_{2,1} V^1_{2,1} > 0. \qquad (III.15.b)$$

Thus, necessary and sufficient conditions obtained from L_s are more strict. However, in L_s one excludes initialy some of the less general perturbations of $\delta\nu$ in this sense analysis through L_s misses some of the necessary and sufficient conditions available in the dynamics. We conclude that L_s and L_w are complementary ways for obtaining necessary and sufficient conditions of stability, for sufficient condition the methods give the same results.

IV. The Energy Criteria for Stability of 2-D Flows

IV.1 Basic Identities of Perturbation Theory

In order to obtain the concrete barotropic flow form of $\delta^2 V$ and $\delta^2 G$ of equations (III.4.b) that are needed for stability analysis, a few basic identities and notations will be given here, this will make our future calculations easier.

We look at the following Eulerian small displacements:

$$\alpha_0(x,y) \to \alpha_0(x,y)+\delta\alpha(x,y),\ \beta_0(x,y) \to \beta_0(x,y)+\delta\beta(x,y),\ \nu_0(x,y) \to \nu_0(x,y)+\delta\nu(x,y) \tag{IV.1.a}$$

and also:

$$\Omega_{c0} \to \Omega_{c0} + \delta\Omega_c, \qquad \vec{b}_0 \to \vec{b}_0 + \delta\vec{b}. \tag{IV.1.b}$$

The subscript $_0$ denotes stationary quantites. It is convenient to introduce the often used $\vec{\xi}$, which is defined by:

$$\alpha(\vec{R}+\vec{\xi}) + \delta\alpha(\vec{R}+\vec{\xi}) \equiv \alpha(\vec{R}) \quad \beta(\vec{R}+\vec{\xi}) + \delta\beta(\vec{R}+\vec{\xi}) \equiv \beta(\vec{R}) \tag{IV.2.a}$$

to order 1,

$$\delta\alpha = -\vec{\xi}\cdot\vec{\nabla}\alpha, \quad \delta\beta = -\vec{\xi}\cdot\vec{\nabla}\beta. \tag{IV.2.b}$$

Having defined $\vec{\xi}$, we can also define the Lagrangian displacement Δ:

$$\Delta \equiv \delta + \vec{\xi}\cdot\vec{\nabla} \tag{IV.3}$$

And ofcourse $\Delta\alpha = \Delta\beta = 0$. The following identities will serve us at future calculations:

$$\Delta\vec{R} = \vec{\xi}, \qquad \delta\vec{\nabla} = \vec{\nabla}\delta, \qquad \Delta\vec{\nabla} = \vec{\nabla}\Delta - \vec{\nabla}\vec{\xi}\cdot\vec{\nabla}, \qquad \Delta\vec{\xi} = 0. \tag{IV.4}$$

The variations of both the surface density Σ and the velocity $\vec{v}$ can be derived from equations (II.5) and (II.11) (for details see paper I see also paper II):

$$\Delta\Sigma = -\Sigma_0\vec{\nabla}\cdot\vec{\xi}, \qquad \delta\Sigma = -\vec{\nabla}\cdot(\Sigma_0\vec{\xi}) \tag{IV.5.a}$$

$$\Delta\vec{v} = -\vec{\nabla}\vec{\xi}\cdot\vec{v}_0 + \vec{\nabla}\Delta\nu, \qquad \delta\vec{v} = -\vec{\nabla}\vec{\xi}\cdot\vec{v}_0 + \vec{\nabla}\Delta\nu - \vec{\xi}\cdot\vec{\nabla}\vec{v}_0, \tag{IV.5.b}$$

we shall also need the second variation of velocity:

$$\Delta^2 \vec{v} = -2\vec{\nabla}\vec{\xi} \cdot \Delta\vec{v} + \vec{\nabla}\Delta^2\nu. \qquad (IV.5.c)$$

IV.2 The Variation of The Potential V

We now give the concrete expression of (III.3) for the potential of a barotropic flow. The potential part of the Lagrangian (II.12) is:

$$V = \int \left[\frac{1}{2}\vec{v}^2 + \varepsilon(\Sigma) + \frac{1}{2}\Phi\right] \Sigma d^2x - \vec{b} \cdot (\vec{P} - \vec{P}_0) - \Omega_c(J - J_0). \qquad (IV.6)$$

Notice that this is also the energy according to equation (III.2).

The condition that $\Delta V = 0$ for arbitrary perturbations of the form (IV.1) is that Euler and the continuity equations of stationary motion are satisfied in moving coordinates as well as the fixation of linear and angular momentum:

$$(\vec{v}_0 - \vec{b}_0 - \Omega_{c0}\vec{1}_z \times \vec{R}) \cdot \vec{\nabla}\vec{v}_0 + \Omega_{c0}\vec{1}_z \times \vec{v}_0 + \vec{\nabla}(h_0 + \Phi_0) = 0, \qquad (IV.7.a)$$

and also:

$$\vec{\nabla} \cdot (\Sigma_0(\vec{v}_0 - \vec{b}_0 - \Omega_{c0}\vec{1}_z \times \vec{R})) = 0, \qquad J = J_0, \qquad \vec{P} = \vec{P}_0. \qquad (IV.7.b)$$

For details see paper I.

We now look at the second variation $\delta^2 V = \Delta^2 V$ at a configuration satisfying equations (IV.7):

$$\Delta^2 V = \int \{(\Delta\vec{v})^2 + \vec{v} \cdot \Delta^2\vec{v} + \Delta[\vec{\xi} \cdot \vec{\nabla}(h+\Phi)]\}|_0 \Sigma_0 d^2x - 2\Delta\vec{b} \cdot \Delta\vec{P} - 2\Delta\Omega_c\Delta J - \vec{b}_0 \cdot \Delta^2\vec{P} - \Omega_{c0}\Delta^2 J,$$

$$(IV.8.a)$$

where:

$$\Delta\vec{P} = \int \Delta\vec{v}\Sigma_0 d^2x, \qquad \Delta^2\vec{P} = \int \Delta^2\vec{v}\Sigma_0 d^2x, \tag{IV.8.b}$$

and:

$$\Delta J = \vec{1}_z \cdot \int (\vec{R}\times\Delta\vec{v}+\vec{\xi}\times\vec{v}_0)\Sigma_0 d^2x, \qquad \Delta^2 J = \vec{1}_z \cdot \int (2\vec{\xi}\times\Delta\vec{v}+\vec{R}\times\Delta^2\vec{v})\Sigma_0 d^2x. \tag{IV.8.c}$$

We demand now that:

$$\Delta J = \Delta\vec{P} = 0 \tag{IV.9}$$

this will constrain the perturbed configuration to have the same linear and angular momentum as the stationary one and will define hopefully both $\Delta\vec{b}$ and $\Delta\Omega_c$. The perturbed potential $\Delta^2 V$ will now have the form:

$$\Delta^2 V = \int \{(\Delta\vec{v})^2 + \vec{v}\cdot\Delta^2\vec{v} + \Delta[\vec{\xi}\cdot\vec{\nabla}(h+\Phi)]\}|_0\Sigma_0 d^2x - \vec{b}_0\cdot\Delta^2\vec{P} - \Omega_{c0}\Delta^2 J, \tag{IV.10}$$

In configuration having special symmetry Ω_{c0} or $\vec{b}_0$ or both, may not be defined by equation (IV.7.b), in this case we will demand: $\Delta^2 J = 0$ or $\Delta^2\vec{P} = 0$ or both. In order to see whether the bilinear form $\Delta^2 V$ is positive around a certain stationary configuration we will have to diagonolize it first this will be done for a few examples in later parts of this paper.

IV.3 The Variation of The Kinetic Term G

Although sufficient conditions of stability can be derived from expression (IV.10) by checking its positivity for certain perturbations, one cannot claim any new knowledge of the stability of the system, unless the perturbation is the most general one. The reason for this is that if $\Delta^2 V > 0$ it may yet be negative for another form of perturbation and hence stability cannot be claimed. If on the other hand $\Delta^2 V < 0$ the system is not unstable

because as we emphasized the $\Delta^2 V > 0$ condition is only sufficient. Thus, in order to achive new information we must check how the perturbation under consideration appears at the kinetic term $\delta^2 G = \Delta^2 G$. As we saw in section III we will have also a necessary condition if the perturbation is of the double mode type. The term G of the Lagrangian (II.12) is of the form:

$$G = \int \left[-\alpha \dot{\beta} - \dot{\nu} \right] \Sigma d^2 x \qquad (IV.11)$$

The first variation of this expression is:

$$\Delta G = \int \left[-\alpha \Delta \dot{\beta} - \Delta \dot{\nu} \right] \Sigma d^2 x, \qquad (IV.12)$$

notice that $\Delta \dot{f} = \dot{(\Delta f)} - \dot{\vec{\xi}} \cdot \vec{\nabla} f$ hence:

$$\Delta G = \int \left[\dot{\vec{\xi}} \cdot \vec{v} - \dot{(\Delta \nu)} \right] \Sigma d^2 x. \qquad (IV.13)$$

For the second variation near equlibrium we obtain:

$$\Delta^2 G = \int \left[\dot{\vec{\xi}} \cdot \Delta \vec{v} + \Delta \dot{\vec{\xi}} \cdot \vec{v} - \Delta \dot{(\Delta \nu)} \right] \Sigma_0 d^2 x. \qquad (IV.14)$$

Using equation (IV.5.b) we obtain:

$$\Delta^2 G = \int \left[2 \dot{\vec{\xi}} \cdot \Delta \vec{v} + \dot{(\Delta^2 \nu)} \right] \Sigma_0 d^2 x. \qquad (IV.15)$$

The last term is a full time derivative and does not contribute to the equations of motion and thus can be neglected, finally we obtain:

$$\Delta^2 G = \int \left[2 \dot{\vec{\xi}} \cdot \Delta \vec{v} \right] \Sigma_0 d^2 x. \qquad (IV.16)$$

V. Applications to Galactic Flows of Uniform Rotation

V.1. Stationary Configurations

The following is taken from Binney and Tremaine (1987) with some changes of notations. We consider disks of fluid rotating with uniform angular velocity $\vec{\Omega} = \vec{1}_z \Omega$ and the velocity field in inertial coordinates is thus:

$$\vec{v}_0 = \vec{\Omega} \times \vec{R} = \Omega R^2 \vec{\nabla} \varphi \tag{V.1}$$

φ is the polar angle, R the radial distance. And the density is:

$$\Sigma_0 = \Sigma_C \tilde{\Sigma}_0(R) \tag{V.2}$$

Inserting equations (V.1) and (V.2) into (IV.7) (taking into account the symmetry of the potential and the enthalpy derived) we see that the variable Ω_{c0} is not defined due to the symmetry of the problem and also:

$$\vec{b}_0 = 0 \tag{V.3}$$

Pressure and specific enthalpy are given by arbitary equations of state:

$$P = P(\Sigma), \quad h = h(\Sigma). \tag{V.4}$$

V.2. Global Properties of Perturbed Configurations.

We now look at the first order perturbation of angular momentum ΔJ of the above configurations as it appears in equation (IV.8.c). We calculate the first term of the above equation and use both equations (V.1) and (IV.5.b):

$$\int \vec{R} \times \Delta \vec{v}|_0 \Sigma_0 d^2x = \int \vec{R} \times (-\vec{\nabla}\vec{\xi} \cdot \vec{v}_0 + \vec{\nabla}\Delta\nu)\Sigma_0 d^2x = -\Omega \int \vec{\xi} \cdot \vec{R}\Sigma_0 d^2x. \tag{V.5}$$

Calculating the second term we obtain:

$$\int \vec{\xi} \times \vec{v}_0 \Sigma_0 d^2x = \int \vec{\xi} \times (\vec{\Omega} \times \vec{R}) \Sigma_0 d^2x = \Omega \int \vec{\xi} \cdot \vec{R} \Sigma_0 d^2x \tag{V.6}$$

Since $(V.5) + (V.6) \equiv 0$, we derive that

$$\Delta J \equiv 0 \tag{V.7}$$

Since Ω_c is not defined we must demand $\Delta^2 J = 0$. Taking $\Delta^2 J = 0$ we are able to obtian differential equality convenient for calculating $\Delta^2 V$. It is obtained from (A.8) in appendix A

$$\Delta^2 V = \int \{\Sigma_0 (\delta \vec{v})^2 + \delta\Sigma \delta(h + \Phi)\} d^2x \tag{V.8}$$

V.3. $\vec{\xi}$ defined by scalar functions

It will appear very convinient to define $\vec{\xi}$ in terms of two independant non dimensional infinitesimal scalars η and ψ as follows:

$$\vec{\xi} = a^2 [\vec{\nabla} \eta + rot \vec{\psi}], \qquad \vec{\psi} = \vec{1}_z \psi. \tag{V.9}$$

a is a scale to be defined later. η is thus defined in terms of $\vec{\xi}$ by

$$\triangle \eta = \frac{1}{a^2} \vec{\nabla} \cdot \vec{\xi} \tag{V.10}$$

Some boundary conditions are needed to make η unique. If we take, say,

$$\eta|_B = 0 \tag{V.11}$$

equation (V.10) has a unique solution. Equation (V.10) represents also the condition of integrability of (V.9), considered as a set of two first order differential equations for ψ, given $\vec{\xi}$ and η. Thus the ψ equations are integrable and define ψ up to a constant.

Following equation (IV.5.b) and equations (V.1) and (V.9), $\delta\vec{v}$ and $\Delta\vec{v}$ may be written as a gradient plus a rotational, always a convenient form for vector fields:

$$\delta\vec{v} = a^2\Omega[\vec{\nabla}\zeta + rot(2\eta\vec{1}_z)] \qquad (V.12.a)$$

$$\Delta\vec{v} = \Omega[\vec{\nabla}(\zeta + \psi) + rot(\eta\vec{1}_z)] \qquad (V.12.b)$$

in which

$$\zeta = \frac{1}{a^2\Omega}(\Delta\nu - \vec{\xi}\cdot\vec{v}_0) - 2\psi \qquad (V.13).$$

VI. Application to a Uniform Rotating Sheet

VI.1. Stationary Configurations

In this section we study a specific case of the flows described in section V. We consider an infinite uniform rotating sheet. Since the density is uniform we have:

$$\tilde{\Sigma} = 1 \qquad (VI.1)$$

There is no self gravitational field in the plane of the sheet, however, to maintain equilibrium we assume the existance of a radial symmetrical external potential:

$$\Phi_0 = \Phi_0(R) \qquad (VI.2)$$

Euler's equations (IV.7.a) relates Ω to $\Phi_0(R)$:

$$\frac{\partial\Phi_0}{\partial R} = \Omega^2 R \qquad (VI.3)$$

VI.2. Fourier Decompositions of $\vec{\xi}$, $\delta\Sigma$, δh, $\delta\Phi$ and ζ

In order to diagonolize the second order perturbation of the potential (V.8) around our present stationary configuration, we will write our perturbed quantities as fourier transforms. First we write η and ψ as a fourier transform:

$$\eta(\vec{R}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta(\vec{k}) e^{i\vec{k}\cdot\vec{R}} d^2k \qquad (VI.4.a)$$

$$\psi(\vec{R}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\vec{k}) e^{i\vec{k}\cdot\vec{R}} d^2k. \qquad (VI.4.b)$$

Next we calculate the Fourier decomposition of $\delta\Sigma$. Using equations (IV.5.a) and equation (V.9), taking the scale $a = 1$, $\delta\Sigma$ can be written as:

$$\delta\Sigma = -\Sigma_C \triangle\eta \qquad (VI.5)$$

and from equations (VI.4) we find that $\delta\Sigma$ has the following expansion:

$$\delta\Sigma = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta\Sigma(\vec{k}) e^{i\vec{k}\cdot\vec{R}} d^2k \qquad (VI.5.a)$$

in which

$$\delta\Sigma(\vec{k}) = \Sigma_C k^2 \eta(\vec{k}) \qquad (VI.5.b)$$

With the expansion of $\delta\Sigma$ we obtain directly the Fourier decomposition of δh:

$$\delta h = \frac{\partial h}{\partial\Sigma}\delta\Sigma \equiv \frac{v_s^2}{\Sigma_C}\frac{1}{2\pi} \int_{-\infty}^{\infty} \delta\Sigma(\vec{k}) e^{i\vec{k}\cdot\vec{R}} d^2k \qquad (VI.6)$$

where v_s is the sound velocity. From the varied Poisson equation we calculate the perturbed gravitational potential:

$$\triangle\delta\Phi = 4\pi G\delta\Sigma\delta_D(z) \qquad (VI.7)$$

where $\delta_D(z)$ is the Dirac function, we find the solution $\delta\Phi$ which has been given by Binney & Tremaine (1987):

$$\delta\Phi = -\int_{-\infty}^{\infty} \delta\Sigma(\vec{k})\frac{2\pi G}{|k|}e^{i\vec{k}\cdot\vec{R}}d^2k. \qquad (VI.8)$$

The only quantity which we nead to decompose in order to obtain the expansion of $\delta\vec{v}$ is ζ which we now write as:

$$\zeta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta(\vec{k})e^{i\vec{k}\cdot\vec{R}}d^2k. \qquad (VI.9)$$

VI.3. Global Constraints

According to equation (IV.9) we must demand that $\Delta\vec{P} = \Delta J = 0$. However, $\Delta J = 0$ for all possible perturbations as we derived in equation (V.7). Thus, we must take care only of $\Delta\vec{P}$. In addition, due to the symmetry of the problem we saw in section V.2 that we must also demand $\Delta^2 J = 0$. replacing equation (V.12.b) into (IV.8.b), we obtain:

$$\Delta\vec{P} = \Sigma_C\Omega\int \vec{\nabla}(\zeta+\psi) + rot(\eta\vec{1}_z)d^2x = 0 \qquad (VI.10)$$

using stokes theorem this can be written as:

$$\Delta\vec{P} = \Sigma_C\Omega\oint[(\zeta+\psi)\vec{1}_z \times d\vec{R} + \eta\vec{1}_z\cdot d\vec{R}] = \Sigma_C\Omega\oint(\zeta+\psi)\vec{1}_z\times d\vec{R} = 0. \qquad (VI.11)$$

In order to fix the linear momenta we take both ψ and ζ to be zero at infinity.

$$\psi = 0, \qquad \zeta = 0 \qquad R\to\infty \qquad (VI.12)$$

The constraint $\Delta^2 J = 0$ becomes, after inserting equations (V.12), (VI.5) and (IV.5.c) into the second part of equation (IV.8.c):

$$\Delta^2 J = -2\Sigma_C\Omega\int \zeta\triangle\frac{\partial\eta}{\partial\varphi}d^2x = 0 \qquad (VI.13)$$

This constraint can be satisfied by taking $\eta(\vec{k}) = \eta(k)$ this will not modify the stability condition which which will obtain in the next section since it depends only on k.

VI.4. Necessary and Sufficient Conditions for Stability of Infinite Sheets

$\Delta^2 V$ is given by (V.8) near the stationary configuration. To obtain the Fourier decomposition of $\Delta^2 V$, we replace $\delta\Sigma$, δh, $\delta\Phi$ and $\delta\vec{v}$ by their respective expansions (VI.5.b), (VI.6), (VI.8) and (VI.9) with (V.12.b), in $\Delta^2 V$ taking account of constraints (V.11), (VI.12) and integrating we obtain the following results:

$$\Delta^2 V = 2\Sigma_C \int k^2[\Omega^2|\zeta|^2 + (4\Omega^2 + k^2 v_s^2 - 2\pi G\Sigma_C|k|)|\eta|^2]d^2k \qquad (VI.14)$$

Following section IV.2 the inequality $\Delta^2 V > 0$, calculable from (VI.14) gives sufficient conditions of stability. From (VI.14) we see that Δ^2 is positive for all k's such that:

$$4\Omega^2 + k^2 v_s^2 - 2\pi G\Sigma_C|k| > 0 \qquad (VI.15.a)$$

this means that the sheet is stable to any wave-length if:

$$\frac{v_s\Omega}{G\Sigma_C} \geq \frac{\pi}{2}. \qquad (VI.15.b)$$

The condition (VI.15) becomes sufficient and necessary when the gyroscopic term for dynamical perturbations $\Delta^2 G$ couples only pairs of modes (see section IV.3).

We can make a Fourier decomposition of $\Delta^2 G$, using $\vec{\xi}$ given by (V.9) and and $\Delta\vec{v}$ given by (V.12.b), and using expansions (VI.4) and (VI.9):

$$\Delta^2 G = 2\Sigma_C\Omega \int k^2\zeta\dot{\eta}^* d^2k + complex\ conjugate. \qquad (VI.16)$$

Necessary and sufficient conditions of stability are obtained from (VI.15) since $\Delta^2 G$ couples only pairs of modes of η and ζ . The condition (VI.15) was previously discovered by Binney

& Tremaine (1987) using the dynamical perturbation method, this was done by our method without solving a single differential equation.

VII. Applications to Maclaurin Disk

VII.1. Stationary Configurations

In this section we study a different case of the flows described in section V. This time we study the stability of of the Maclaurin disk which is a finite uniform rotating disk. The density of this disk is given by:

$$\tilde{\Sigma} = \sqrt{1 - \frac{R^2}{R_D^2}} \equiv \chi \qquad R \leq R_D, \tag{VII.1.a}$$

$$\tilde{\Sigma} = 0 \qquad\qquad R \geq R_D. \tag{VII.1.b}$$

R_D is the radius of the disk. We shall take $a = R_D$. The self gravitational field in the plane of the disk is:

$$\Phi_0 = \frac{1}{2}\Omega_0^2 R^2 + const., \qquad \Omega_0^2 = \frac{\pi^2 G \sigma_C}{2a} \tag{VII.2}$$

Ω_0 is the angular velocity of a test particle on a circular orbit. See Binney and Tremaine (1987) for further details. Here we allow only a certain equation of state such that the Pressure and specific enthalpy are given by:

$$P = \kappa \Sigma^3 \tag{VII.3.a}$$

$$h = \frac{3}{2}\kappa \Sigma^2 \tag{VII.3.b}$$

Euler's equations relate Ω to Ω_0, Σ_C and a:

$$\Omega^2 = \Omega_0^2 - \frac{3\kappa\Sigma_C^2}{a^2} \tag{VII.4}$$

VII.2. Spherical Harmonic Decompositions of $\vec{\xi}$, $\delta\Sigma$, δh, $\delta\Phi$ and ζ

We now decompose η and ψ in normalised spherical harmonic functions of χ defined in equation (VII.1) and φ. We have:

$$\eta = \sum_{l=m}^{\infty}\sum_{m=0}^{\infty} \eta_{lm}\mathcal{P}_l^m(\chi)e^{im\varphi} + c.c. \qquad (VII.5.a)$$

$$\psi = \sum_{l=m}^{\infty}\sum_{m=0}^{\infty} \psi_{lm}\mathcal{P}_l^m(\chi)e^{im\varphi} + c.c. \qquad (VII.5.b)$$

η_{lm} and ψ_{lm} are arbitary, infinitesimal, complex numbers. Associated Legendre polynomials are defined in the range $-1 \leq \chi \leq 1$, but η and ψ are only defined in the range $0 \leq \chi \leq 1$. We may extend η, ψ to the negative region of χ in any way we want. However, these functions have bounded gradients at $\chi = 0$. Indeed, since

$$|\partial_R\eta| < \infty, \qquad |\partial_R\psi| < \infty \qquad (VII.6.a)$$

$$|\frac{1}{R}\partial_\varphi\eta| < \infty, \qquad |\frac{1}{R}\partial_\varphi\psi| < \infty \qquad (VII.6.b)$$

hold and since

$$\partial_R = -\frac{1}{a\chi}\sqrt{1-\chi^2}\partial_\chi, \qquad (VII.7)$$

then,

$$\partial_\chi\eta|_{\chi=0} = \partial_\chi\psi|_{\chi=0} = 0. \qquad (VII.8)$$

Equations (VII.8) will be satisfied if we make symmetrical continuous extensions

$$\eta(\chi) = \eta(-\chi), \qquad (VII.9.a)$$

$$\psi(\chi) = \psi(-\chi), \qquad (VII.9.b)$$

whose expansions are given by (VII.5) with $(l-m)$ even. The expansions in the domain $0 \leq \chi \leq 1$, of *arbitrary* continuous η and ψ that satisfy (VII.6) everywhere are then given by (VII.5) with $(l-m)$ even [Arfken (1985)] . The boundary conditions (V.11) gives the following relations among the η_{lm}'s: for every $m \geq 0$:

$$\sum_{l=m}^{\infty} \eta_{lm} \mathcal{P}_l^m(0) = 0, \qquad l-m \;\; even \qquad (VII.10)$$

Equations (VII.10) define, say, η_{ll} in terms of $\eta_{l,m\neq l}$.

We can now calculate the spherical harmonic decomposition of $\delta\Sigma$ using equations (IV.5.a) and (V.9):

$$\delta\Sigma = -a^2 \Sigma_C [\vec{\nabla} \cdot (\chi \vec{\nabla} \eta) + \vec{\nabla} \chi \times \vec{\nabla} \psi]. \qquad (VII.11)$$

and from equation (VII.11) and (VII.5) we find that $\delta\Sigma$ has the following expansion:

$$\delta\Sigma = \Sigma_C \sum_{l=m}^{\infty} \sum_{m=0}^{\infty} \Sigma_{lm} \frac{\mathcal{P}_l^m(\chi)}{\chi} e^{im\varphi} + c.c. \qquad l-m \; even \qquad (VII.12.a)$$

in which

$$\Sigma_{lm} = [l(l+1) - m^2]\eta_{lm} + im\psi_{lm} \equiv k_{lm}\eta_{lm} + im\psi_{lm} \qquad (VII.12.b)$$

With the expansion of $\delta\Sigma$ we obtain directly through (VII.3.b) the spherical harmonic decomposition of δh:

$$\delta h = 3\kappa \Sigma_C^2 \sum_{l=m}^{\infty} \sum_{m=0}^{\infty} \Sigma_{lm} \mathcal{P}_l^m(\chi) e^{im\varphi} + c.c. \qquad l-m \; even \qquad (VII.13)$$

and from the varied Poisson equation :

$$\triangle \delta\Phi = 4\pi G \delta\Sigma \delta_D(z) \qquad (VII.14)$$

where $\delta_D(z)$ is the Dirac function, we find the solution $\delta\Phi$ which has been given by Hunter (1963):

$$\delta\Phi = \Omega_0^2 a^2 \sum_{l=m}^{\infty} \sum_{m=0}^{\infty} \Phi_{lm} \mathcal{P}_l^m(\chi) e^{im\varphi} + c.c. \qquad l-m \ even \qquad (VII.15.a)$$

in which

$$\Phi_{lm} = -g_{lm} \Sigma_{lm} \qquad (VII.15.b)$$

where

$$g_{lm} = \frac{(l+m)!(l-m)!}{2^{2l-1}[(\frac{l+m}{2})!(\frac{l-m}{2})!]^2} < 1 \qquad l > 0, \qquad l-m \ even \qquad (VII.15.c)$$

We now expand ζ in spherical harmonics:

$$\zeta = \sum_{l=m}^{\infty} \sum_{m=0}^{\infty} \zeta_{lm} \mathcal{P}_l^m(\chi) e^{im\varphi} + c.c., \qquad l-m \ even. \qquad (VII.16)$$

VII.3. Global Constraints

In this section as in VI.3 we must take care of $\Delta\vec{P} = 0$ and $\Delta^2 J = 0$. Inserting equation (V.12.b) into the first expression of (IV.8.b) and using expansions (VII.5) and (VII.16) we obtain the result:

$$\zeta_{11} = -\psi_{11} - i\eta_{11}. \qquad (VII.17)$$

i.e. in order to fix the momentum we must take ζ_{11} to be a dependent variable. The constraint $\Delta^2 J = 0$ becomes, (the calculation is in appendix B):

$$\Delta^2 J = 4\pi\Sigma_C a^4 \Omega \sum m[i\zeta_{lm}\Sigma*_{lm} + m(|\psi_{lm}|^2 - |\eta_{lm}|^2)] + c.c. = 0 \qquad (VII.18)$$

VII.4. Necessary and Sufficient Conditions for Stability of Maclaurin Disks

In order to study the stability features of the Maclaurin Disks we shall have to diagonolize $\Delta^2 V$ given by (V.8). To obtain the spherical harmonic decomposition of $\Delta^2 V$, we replace $\delta\Sigma$, δh, $\delta\Phi$ by their respective expansions (VII.12), (VII.13) and (VII.15). Also we write $\delta\vec{v}$ by the represantation (V.12.a) using the expansions (VII.5.a), (VII.16) . The result is as the follows [detailed calculations are given in appendix C]:

$$\Delta^2 V = 4\pi\Sigma_C\Omega^2 a^4 \sum_{(l=m)>1,m=0}^{\infty} 4(k_{lm} - \frac{m^2}{k_{lm}})|\eta_{lm}|^2 + k_{lm}|\zeta_{lm} + \frac{2im}{k_{lm}}\eta_{lm}|^2 +$$
$$[-1 + (1 - g_{lm})\frac{\Omega_0^2}{\Omega^2}]|\Sigma_{lm}|^2 \qquad (VII.19)$$

where we have taken into account the constraint (VII.17). Following section IV.2 the inequality $\Delta^2 V > 0$, calculable from (VI.19) with the constraint (VII.18), gives sufficient conditions of stability. $\Delta^2 V$ has the form $\frac{\Omega_0^2}{\Omega^2}A^2 - B > 0$. If $B < 0$, the disk is stable for any $\frac{\Omega^2}{\Omega_0^2}$. For instance, Maclaurin disks are stable to any perturbation that keeps the same densities at the displaced points ($\delta\Sigma = 0$). We are naturally interested in perturbations that might upset stability and for which $B > 0$. If $B > 0$ then $\frac{\Omega^2}{\Omega_0^2}$ must satisfy the following inequality

$$Q \equiv \frac{\Omega^2}{\Omega_0^2} < \frac{\sum_{(l=m)>1,m=0}^{\infty}(1 - g_{lm})|\Sigma_{lm}|^2}{\sum_{(l=m)>1,m=0}^{\infty}|\Sigma_{lm}|^2 - 4(k_{lm} - \frac{m^2}{k_{lm}})|\eta_{lm}|^2 - k_{lm}|\zeta_{lm} + \frac{2im}{k_{lm}}\eta_{lm}|^2} \qquad (VII.20)$$

The condition (VII.20) supplied by the constraint (VII.18) becomes sufficient and necessary when the gyroscopic term for dynamical perturbations $\Delta^2 G$ couple only two modes (see section III). It is therefore important to calculate $\Delta^2 G$. We can make a spherical harmonic decomposition of $\Delta^2 G$ given in equation (IV.16). A straightforward subtitution of $\vec{\xi}$ given

by equations (V.9) and (VII.5) and $\Delta\vec{v}$ given by equations (V.12.b), (VII.5) and (VII.16) leads to the following expression for $\Delta^2 G$:

$$\Delta^2 G = 4\pi\Sigma_C \Omega a^4 \sum_{l,m\neq 1,1} [\dot{\Sigma}_{lm}\zeta^*_{lm} + im(\dot{\eta}^*_{lm}\eta_{lm} - \dot{\psi}^*_{lm}\psi_{lm})] + complex\ conjugate \quad (VII.21)$$

in which (VII.17) and (VII.18) has to be taken into account. *Necessary and sufficient conditions of stability are obtained from (VII.20) when $\Delta^2 G$ couples pairs only.*

VII.5. Symmetric and Antisymmetric Single-Mode Perturbations

In a single-mode analysis condition (VII.18) reduces to:

$$m[i\zeta_{lm}\Sigma *_{lm} + m(|\psi_{lm}|^2 - |\eta_{lm}|^2)] + c.c. = 0 \quad (VII.22)$$

For symmetrical and antisymmetrical modes, we take either real or imaginary components of η_{lm} and $i\psi_{lm}$ in the spherical harmonic expansion, so the Fourier expansion contains either $cos(m\varphi)$ or $sin(m\varphi)$. For such perturbations, $\Delta^2 G$ becomes:

$$\Delta^2 G = 4\pi\Sigma_C \Omega a^4 \sum_{l,m\neq 1,1} [\dot{\Sigma}_{lm}\zeta^*_{lm}] + complex\ conjugate \quad (VII.23)$$

where we have taken into account the constraint (VII.17). Notice that we have used without loss of generality we assume that Σ_{lm} is real (and so is η_{lm} and $i\psi_{lm}$), and equation (VII.23) becomes:

$$\Delta^2 G = 8\pi\Sigma_C \Omega a^4 \sum_{l,m\neq 1,1} [\dot{\Sigma}_{lm}\zeta_{lmR}]. \quad (VII.24)$$

We now have a pair coupling of Σ_{lm} and ζ_{lmR}, hence we can achive a necessary and sufficient condition of stability. The imiginary part of ζ_{lm} can now be obtained through

equation (VII.22), assuming non-radial modes (for radial modes equation (VII.22) is trivially satisfied):

$$\zeta_{lmI} = m\frac{\psi_{lm}^2 - \eta_{lm}^2}{\Sigma_{lm}} \qquad (VII.25)$$

It is advantageous to introduce the following notations:

$$z \equiv -k_{lm}\frac{\eta_{lm}}{\Sigma_{lm}} \qquad -\infty < z < \infty \qquad (VII.26.a)$$

and

$$x \equiv \frac{m^2}{k_{lm}^2} \qquad 0 \leq x \leq 1. \qquad (VII.26.b)$$

For one single mode (l, m), equation (VII.20) with equation (VII.25) reduces to in term of z:

$$Q < Q_{lm} = \frac{(1 - g_{lm})}{P_4(z)} \qquad (VII.27.a)$$

we have assumed that ζ_{lmR} is equal to zero since any other value of ζ_{lmR} will make Q_{lm} bigger, and we are intrested only in its lowest value. $P_4(z)$ is a polynomial of order 4 in z,

$$P_4(z) = 1 - \frac{1}{k_{lm}}\{(\frac{1}{x} - 1)[(1-x)z^4 + 4(1-x)z^3 + 6z^2 + 4z] + \frac{1}{x}\} \qquad (VII.27.b)$$

$P_4(z)$ has one and only one real maximum for any x. We are only interested in values of z for which $P_4(z) > 0$. For $P_4(z) < 0$, $\Delta^2 V > 0$ for any value of Q. The maximum of $P_4(z)$ is obtained for:

$$z_{max}(x) = \frac{x^{\frac{1}{3}}}{\sqrt{1-x}}[(\sqrt{1-x} - 1)^{\frac{1}{3}} + (\sqrt{1-x} + 1)^{\frac{1}{3}}] - 1 \qquad (VII.28.a)$$

for which

$$P_4(z)_{max} = 1 - \frac{1}{k_{lm}}\{(\frac{1}{x} - 1)[(1-x)z_{max}^4 + 4(1-x)z_{max}^3 + 6z_{max}^2 + 4z_{max}] + \frac{1}{x}\} \equiv 1 - \frac{y(x)}{k_{lm}} \qquad (VII.28.b)$$

To any pair of values (l, m) corresponds a value x defined by (VII.26.b) and a value $y(x)$. Following (VII.27.a) and (VII.28), we must thus have

$$Q < (Q_{lm})_{min} = \frac{(1 - g_{lm})}{1 - \frac{y(x)}{k_{lm}}} \qquad (VII.29)$$

The smallest minimum of Q_{lm} is obtained for $(l, m) = (2, 2)$ for which $(Q_{22})_{min} = \frac{1}{2}$. Therefore the necessary and sufficient condition of stability with respect to symmetric or antisymmetric single mode perturbations of Maclaurin disks is

$$Q < \frac{1}{2} \qquad (VII.30)$$

VII.6. Comments

a) Binney and Tremaine (1987) have given the following dispersion relation for the ω - modes of dynamical perturbations:

$$\omega_r^3 - \omega_r\{4\Omega^2 + k_{lm}[\Omega_0^2(1 - g_{lm}) - \Omega^2]\} + 2m\Omega[\Omega_0^2(1 - g_{lm}) - \Omega^2] = 0, \qquad \omega_r = \omega - m\Omega \qquad (VII.31)$$

Stability holds if the non spurious ω_r's are real roots. Equation (VII.31) is a third order polynome. The condition for a polynome of the form $x^3 + a_2 x + a_3 = 0$ to have only real roots is given in standard handbooks [for instance: Schaum's Mathematical Handbook (1968)]

$$(\frac{a_2}{3})^3 + (\frac{a_3}{2})^2 < 0 \qquad (VII.32)$$

Inequality (VII.32) for equation (VII.31) is exactly our inequality (VII.29). The results of Binney & Tremaine were obtained by solving three couple differential equations, our results were obtained without **solving a single equation**.

b) For radial modes, $m = 0$, , $\Delta^2 J \equiv 0$ and

$$\Delta^2 G = 4\pi \Sigma_C \Omega a^4 \sum_{l=0} [\dot{\Sigma}_{l0} \zeta_{l0}]. \qquad (VII.33)$$

that is all radial modes are only pair coupled. Moreover, since [see equation (VII.12.b)]

$$\Sigma_{l0} = l(l+1)\eta_{l0} \qquad (VII.34)$$

all l-modes are decoupled, $\Delta^2 V$ is a sum of squares of independent modes. In this case one obtains stability limits for non single modes. The most unfavorable limit, however, is again that given by Q_{l0} for which $y = 4$ and $k_{ll} = l(l+1)$:

$$Q < Q_{l0} = \frac{1 - g_{l0}}{1 - \frac{4}{l(l+1)}} \qquad (l \geq 2), \qquad g_{l0} = \frac{l!^2}{2^{2l-1}[(\frac{l}{2})!]^4} \qquad (VII.35)$$

c) Notice the similarity of the mathematical analysis in the weak energy method and the strong method of paper II.

VIII. Application to a Rayleigh's Shear Flow

VIII.1. Stationary Configurations

The problem of the stability of a shear flow was considered by most developers of energy methods such as Arnold (1969), Holm at al. (1983) and Grinfeld (1984). It interesting therefore to compare it to our own weak energy method. We consider a flow in a rectengular pipe, which occupies the domain: $[0 - L, 0 - 1]$ that is the rigid walls of the pipe are the lines $y = 0,\ y = 1$ while the lines $x = 0,\ x = L$ are identified, in this way we construct a "torus". x, y are the Caretesian coordinates. We define the flow such that:

$$\vec{v}_0 = W(y)\vec{1}_x. \qquad (VIII.1)$$

The uniform density is set to unity

$$\Sigma_0 = 1 \qquad (VIII.2)$$

Pressure and specific enthalpy are given by arbitary equations of state

$$P = P(\Sigma) \qquad (VIII.3.a)$$

$$h = h(\Sigma) \qquad (VIII.3.b)$$

There is no potential of any kind in this problem. Inserting equations (VIII.1-3) into the equations of motion (IV.7) we obtain the following:

$$\Omega_c = 0, \qquad b_{0y} = 0 \qquad (VIII.4)$$

b_{0x} is left undefined as was Ω_c in the models considered before. Notice in this problem we do not have a "free boundary" but a fixed one, this modifies the possible perturbations one can construct such that:

$$\delta\nu(0,y) = \delta\nu(L,y), \qquad \vec{\xi}(0,y) = \vec{\xi}(L,y) \qquad (VIII.5.a)$$

$$\xi_y(x,0) = \xi_y(x,1) = 0 \qquad (VIII.5.b)$$

since we consider here a torus with rigid boundaries.

VIII.2. Global Constraints

In this case $\Omega_c = 0,\ b_{0y} = 0$ are defined but not b_{0x}. This implies that not only do we have to fix: $\Delta\vec{P} = 0,\ \Delta J = 0$ but $\Delta^2 P_x = 0$ as well. Inserting equations (VIII.1), (VIII.2) and (IV.5.b) in the first equation of (IV.8.b) we obtain:

$$\Delta\vec{P} = \int \Delta\vec{v} d^2x = \int (-\vec{\nabla}\xi_x W + \vec{\nabla}\Delta\nu) d^2x = \vec{1}_y \int -\frac{\partial\xi_x}{\partial y} W + \frac{\partial\Delta\nu}{\partial y} d^2x = 0 \quad (VIII.6.a)$$

Notice that the x-component which contained only boundary terms vanished do to equations (VIII.5). The second order $\Delta^2 P_x = 0$ becomes using equation (IV.5.c)

$$\Delta^2 P_x = \int \Delta^2 v_x d^2x = -2\int \frac{\partial \vec{\xi}}{\partial x} \cdot \Delta \vec{v} + \frac{\partial \Delta^2 \nu}{\partial x} d^2x = -2\int \frac{\partial \vec{\xi}}{\partial x} \cdot \Delta \vec{v} d^2x \qquad (VIII.6.b)$$

$\Delta^2 \nu$ term vanishes due to integration by parts. The angular momentum is fixed by:

$$\Delta J = \vec{1}_z \cdot \int \vec{\xi} \times \vec{v}_0 + \vec{R} \times \Delta \vec{v} d^2x = \int -\xi_y W + \vec{1}_z \cdot (\vec{R} \times \Delta \vec{v}) d^2x \qquad (VIII.6.c)$$

VIII.3. $\Delta^2 V$

We start from equation (IV.8.a). Using equations (A.6), (VIII.3) we have:

$$\Delta(\vec{\xi} \cdot \vec{\nabla} h) = \vec{\xi} \cdot \vec{\nabla} \delta h = \vec{\xi} \cdot \vec{\nabla}(\frac{\partial h}{\partial \Sigma}|_0 \delta \Sigma) = \vec{\xi} \cdot \vec{\nabla}(v_s^2 \delta \Sigma) \qquad (VIII.7)$$

we also used the definition of the velocity of sound v_s. This means that the "potential" part of equation (IV.8.a) becomes using equations (VIII.5), (IV.5.a) and integrating by parts:

$$\Delta^2 V_p = \int \Delta[\vec{\xi} \cdot \vec{\nabla} h] d^2x = \int v_s^2 \delta \Sigma^2 d^2x \qquad (VIII.7)$$

Using equation (IV.3) and (VIII.1) we have:

$$\Delta \vec{v} = \delta \vec{v} + \xi_y W' \vec{1}_x. \qquad (VIII.8)$$

Putting equations (VIII.8), (IV.5.c) into the "kinetic" part of equation (IV.8.a) we have:

$$\Delta^2 V_k = \int [\delta \vec{v}^2 + 2\xi_y W' \delta v_x + \xi_y^2 W'^2 - 2W \frac{\partial \vec{\xi}}{\partial x} \cdot (\delta \vec{v} + \xi_y W' \vec{1}_x)] d^2x \qquad (VIII.8)$$

$\Delta^2 \nu$ term vanishes due to integration by parts. We use equation (IV.5.a) to subtitute $\frac{\partial \xi_x}{\partial x} = -\delta \Sigma - \frac{\partial \xi_y}{\partial y}$ in equation (VIII.8) and obtain:

$$\Delta^2 V_k = \int [\delta \vec{v}^2 + 2\xi_y W' \delta v_x + \xi_y^2 W'^2 + 2W(W'\xi_y + \delta v_x)(\delta \Sigma + \frac{\partial \xi_y}{\partial y}) - 2W \frac{\partial \xi_y}{\partial x} \delta v_y] d^2x \qquad (VIII.9)$$

Integrating by parts the second and last terms and collecting some terms together we arrive at the following form:

$$\Delta^2 V_k = \int [\delta v_y^2 + (\delta v_x + W\delta\Sigma)^2 - W^2\delta\Sigma^2$$
$$+ 2W\xi_y(\frac{\partial \delta v_y}{\partial x} - \frac{\partial \delta v_x}{\partial y}) + 2WW'\xi_y(\delta\Sigma + \frac{\partial \xi_y}{\partial y}) + \xi_y^2 W'^2] d^2x \qquad (VIII.10)$$

The term $\delta\omega = (\frac{\partial \delta v_y}{\partial x} - \frac{\partial \delta v_x}{\partial y})$ which is the variation of the vorticity can be written using equations (IV.5) and (VIII.1) as:

$$\delta\omega = \xi_y W'' - \delta\Sigma W' \qquad (VIII.11)$$

Inserting equation (VIII.11) into (VIII.10) and canceling the apropriate terms we arrive at the simpler expression:

$$\Delta^2 V_k = \int [\delta v_y^2 + (\delta v_x + W\delta\Sigma)^2 - W^2\delta\Sigma^2 + WW''\xi_y^2] d^2x. \qquad (VIII.12)$$

Thus:

$$\Delta^2 V = \Delta^2 V_k + \Delta^2 V_p = \int [\delta v_y^2 + (\delta v_x + W\delta\Sigma)^2 + (v_s^2 - W^2)\delta\Sigma^2 + WW''\xi_y^2] d^2x. \quad (VIII.13)$$

From this we extract the following famous sufficient condition:

$$v_s^2 > W^2 \qquad and \qquad WW'' > 0. \qquad (VIII.14)$$

This was obtained by Lord Rayleigh (1880), Arnold (1966), Holm et al. (1983) and Grinfeld (1984). This is obiously a weak condition since we did not take full advantage of the constraints (VIII.6).

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Appendix A: $\Delta^2 V$ Variational Identity (V.8)

We start from (IV.10). Inserting equation (V.1) into equation (IV.10) we obtain:

$$\Delta^2 V = \int \{(\Delta\vec{v})^2 + (\vec{\Omega}\times\vec{R})\cdot\Delta^2\vec{v} + \Delta[\vec{\xi}\cdot\vec{\nabla}(h+\Phi)]\}|_0 \Sigma_0 d^2x \qquad (A.1)$$

Since we demand that $\Delta^2 J = 0$, we see from the second part of (IV.8.c) that we can get read of the $\Delta^2\vec{v}$ term thus obtaining:

$$\Delta^2 V = \int \{(\Delta\vec{v})^2 - \vec{\Omega}\cdot(2\vec{\xi}\times\Delta\vec{v}) + \Delta[\vec{\xi}\cdot\vec{\nabla}(h+\Phi)]\}|_0 \Sigma_0 d^2x \qquad (A.2)$$

The terms containing $\Delta\vec{v}$ can be written as a difference of squares:

$$(\Delta\vec{v})^2 - \vec{\Omega}\cdot(2\vec{\xi}\times\Delta\vec{v}) = (\Delta\vec{v}-\vec{\Omega}\times\vec{\xi})^2 - (\vec{\Omega}\times\vec{\xi})^2 = (\Delta\vec{v}-\vec{\xi}\cdot\vec{\nabla}\vec{v}_0)^2 - \Omega^2\vec{\xi}^2 = (\delta\vec{v})^2 - \Omega^2\vec{\xi}^2 \quad (A.3)$$

In the last equality we have used the second part of equation (IV.5.b). So equation (A.2) with (A.3) can now be written as:

$$\Delta^2 V = \int \{(\delta\vec{v})^2 - \Omega^2\vec{\xi}^2 + \Delta[\vec{\xi}\cdot\vec{\nabla}(h+\Phi)]\}|_0 \Sigma_0 d^2x \qquad (A.4)$$

Using the third operator identity of equation (IV.4), we may rewrite the $h+\Phi$ term in (A.1) as follows:

$$\Delta[\vec{\xi}\cdot\vec{\nabla}(h+\Phi)] = \vec{\xi}\cdot\vec{\nabla}\Delta(h+\Phi) - \vec{\xi}\cdot\vec{\nabla}\vec{\xi}\cdot\vec{\nabla}(h+\Phi) = \vec{\xi}\cdot\vec{\nabla}\delta(h+\Phi) + \vec{\xi}\cdot\vec{\nabla}\vec{\nabla}(h+\Phi)\cdot\vec{\xi} \quad (A.6)$$

But from Euler's equations (IV.7.a) we see that

$$\vec{\nabla}(h+\Phi)|_0 = -\vec{v}_0\cdot\vec{\nabla}\vec{v}_0 \qquad (A.7)$$

Subtituting (A.7) into (A.6) and $\Delta[\vec{\xi}\cdot\vec{\nabla}(h+\Phi)]$ back in (A.4), we find that $\Omega^2\vec{\xi}^2$ cancels out. Finally with $\delta\Sigma$ given in (IV.5.a), and with some integration by parts, we obtain the following much simpler form for $\Delta^2 V$ which is that written in (V.8):

$$\Delta^2 V = \int \{\Sigma(\delta\vec{v})^2 + \delta\Sigma\delta(h+\Phi)\}|_0 d^2x \qquad (A.8)$$

Appendix B: Spherical Harmonic Decomposition of $\Delta^2 J$

Following the second part of (IV.8.c):

$$\Delta^2 J = \vec{1}_z \cdot \int (2\vec{\xi} \times \Delta\vec{v} + \vec{R} \times \Delta^2 \vec{v})|_0 \Sigma_0 d^2x. \qquad (B.1)$$

Inserting equation (IV.5.c) into (B.1) we obtain:

$$\Delta^2 J = \vec{1}_z \cdot \int (2\vec{\xi} \times \Delta\vec{v} - 2\vec{R} \times \vec{\nabla}\vec{\xi} \cdot \Delta\vec{v})|_0 \Sigma_0 d^2x. \qquad (B.2)$$

Inserting (V.12.b) and (V.1) into (B.2) gives

$$\Delta^2 J = 2a^4 \Omega \vec{1}_z \cdot \int \{[\vec{\nabla}\eta + rot\vec{\psi}] \times [\vec{\nabla}(\zeta + \psi) + rot(\eta\vec{1}_z)]$$
$$- \vec{R} \times \vec{\nabla}[\vec{\nabla}\eta + rot\vec{\psi}] \cdot [\vec{\nabla}(\zeta + \psi) + rot(\eta\vec{1}_z)]\} \Sigma_0 d^2x. \qquad (B.3)$$

The spherical harmonic decomposition of $\Delta^2 J$ is obtained by replacing in equation (B.3) ζ by (VII.16), Σ_0 by $\Sigma_C \chi$ (equation (VII.1)) and η, ψ by (VII.5). Using also (VII.12.b) for Σ_{lm}, the result comes out as follows:

$$\Delta^2 J = 4\pi \Sigma_C a^4 \Omega \sum m[i\zeta_{lm} \Sigma *_{lm} + m(|\psi_{lm}|^2 - |\eta_{lm}|^2)] + c.c. \qquad (B.4)$$

and demanding $\Delta^2 J = 0$ we obtain equation (VII.17).

Appendix C: Spherical Harmonic Decomposition of $\Delta^2 E$

Let us start from (A.8) and take $\Delta^2 J = 0$. We shall set

$$\Delta^2 V = \Delta^2 V_k + \Delta^2 V_p \qquad (C.1.a)$$

in which

$$\Delta^2 V_k = \int [\Sigma(\delta\vec{v})^2]|_0 d^2x \qquad (C.1.b)$$

and

$$\Delta^2 V_p = \int [\delta\Sigma(\delta h + \delta\Phi)]|_0 d^2x. \tag{C.1.c}$$

Inserting (V.12.a) in (C.1.b) gives:

$$\Delta^2 V_k = a^4\Omega^2 \int \{(\vec{\nabla}\zeta)^2 + 2\vec{\nabla}\cdot[\zeta rot(2\eta\vec{1}_z)] + [rot(2\eta\vec{1}_z)]^2\}\Sigma_0 d^2x \tag{C.2}$$

Using Gauss's theorem, we can integrate (C.2) by part, and with $\Sigma_0|_B = 0$, obtain:

$$\Delta^2 V_k = a^4\Omega^2 \int [-\zeta\vec{\nabla}\cdot(\Sigma_0\vec{\nabla}\zeta) - 2\zeta\vec{\nabla}\Sigma_0 \cdot rot(2\eta\vec{1}_z) - 4\eta\vec{\nabla}\cdot(\Sigma_0\vec{\nabla}\eta)]d^2x. \tag{C.3}$$

The spherical harmonic decomposition of $\Delta^2 V_k$ is obtained by replacing in equation (C.3), ζ by (VII.16), Σ_0 by $\Sigma_C\chi$ (equation (VII.1)) and η by (VII.5.a). The result comes out as follows:

$$\Delta^2 V_k = 4\pi\Sigma_C\Omega^2 a^4 \sum_{l=m,m=0}^{\infty} [4(k_{lm} - \frac{m^2}{k_{lm}})|\eta_{lm}|^2 + k_{lm}|\zeta_{lm} + \frac{2im}{k_{lm}}\eta_{lm}|^2] \tag{C.4}$$

Notice that these are already some of the terms of $\Delta^2 V$ given in (VII.18)

The spherical harmonic decomposition of $\Delta^2 V_p$ follows by inserting in (C.1.c) the respective expansions of $\delta\Sigma$ in (VII.12), δh in (VII.13) and $\delta\Phi$ in (VII.15). For $\Delta^2 V_p$, we obtain :

$$\Delta^2 V_p = 2\pi a^4\Omega_0^2 \sum_{l=m}^{\infty}\sum_{m=0}^{\infty} \Sigma_{lm}[\Phi_{lm}^* + 3\kappa\Sigma_C\Sigma_{lm}^*] + c.c. \tag{C.5}$$

With (VII.15.b) and (VII.4), $\Delta^2 V_p$ can also be written:

$$\Delta^2 V_p = 4\pi\Sigma_C a^4\Omega^2 \sum_{l=m,m=0}^{\infty} [-1 + (1 - g_{lm})\frac{\Omega_0^2}{\Omega^2}]|\Sigma_{lm}|^2 \tag{C.6}$$

The sum of $\Delta^2 V_k$ given by (C.4) and $\Delta^2 E_p$ of (C.6) is the expression of $\Delta^2 v$ written in (VII.18).